A METHOD OF CALCULATING THE TEMPERATURE FIELD IN MULTILAYER MEDIA

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UDC 536.24

An approximate method is developed for calculating the temperature field in composite bodies for arbitrary boundary conditions on the outer and inner boundaries.

A large number of thermophysical problems, related to the study of the formation process of temperature fields in complicated and inhomogeneous systems, reduces to calculating the nonstationary field in multilayer media [1, 2]. As shown in [3], the temperature field in these systems depends strongly on the value of the contact resistance of adjacent layers, account of which is needed for a large number of technical instruments.

Presently there exists a number of general methods of obtaining analytic solutions in multilayer media in the presence of an ideal contact between layers [4-10]. A critical use of the available methods, however, is accompanied by significant mathematical difficulties, sharply increasing with the number of components of system layers. These are due to the necessity of solving in each specific case an algebraic system of equations, whose number equals twice the number of layers. The solution of the multilayer problem for nonideal contact between layers is reduced in [11, 12] to a system of integral equations, which also involves certain mathematical difficulties with increasing number of layers.

In the present formulation the mathematical problem reduces to solving a system of equations in partial derivatives

$$a_k \frac{\partial^2 U_k}{\partial x^2} = \frac{\partial U_k}{\partial t}; \quad H_{k-1} \leqslant x \leqslant H_k \quad (k = 1, 2, \ldots, n) \tag{1}$$

with the following boundary conditions:

$$\lambda_1 \frac{\partial U_1}{\partial x} - \alpha_1(t) U_1 = -f_1(t) \quad \text{for} \quad x = 0, \tag{2}$$

$$\lambda_k \frac{\partial U_k}{\partial x} = \lambda_{k+1} \frac{\partial U_{k+1}}{\partial x} \quad \text{for} \quad x = H_k, \tag{3}$$

$$R_k \lambda_k \frac{\partial U_k}{\partial x} = U_k - U_{k+1} \quad (k = 1, 2, \ldots, n-1),$$

$$\lambda_n \frac{\partial U_n}{\partial x} + \alpha_2(t) U_n = f_2(t) \quad \text{for} \quad x = H_n \tag{4}$$

and initial conditions

$$U_k = \varphi_k(x) \quad (k = 1, 2, \ldots, n) \quad \text{for} \quad t = 0. \tag{5}$$

Let a solution of the problem (1)-(5) exist and be unique. There exist then unique functions $\psi_k^\gamma(t)$ ($\gamma = 1, 2$), satisfying the conditions:

$$\psi_k^1(t) = U_k(H_{k-1}, t), \quad \psi_k^2(t) = U_k(H_k, t). \tag{6}$$

We denote by $G_k(x, t, \xi, \tau)$ the Green's function for the Cauchy problem in the region $[H_{k-1}, H_k]$. The solution for the k-th layer, satisfying the k-th equation and the k-th initial condition, is then [13]

$$U_k(x, t) = a_k \int_0^t \psi_k^1(\tau) \frac{\partial}{\partial \xi} G_k(x, t, H_{k-1}, \tau)\, d\tau - a_k \int_0^t \psi_k^2(\tau) \frac{\partial}{\partial \xi} G_k(x, t, H_k, \tau)\, d\tau + D_k(x, t), \tag{7}$$

Agrophysical Scientific-Research Institute, Leningrad. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 43, No. 3, pp. 483-492, September, 1982. Original article submitted June 5, 1981.

 0022-0841/82/4303-1048$07.50

where

$$D_k(x, t) = \int_{H_{k-1}}^{H_k} \varphi_k(\xi) G_k(x, t, \xi, 0)\, d\xi. \tag{8}$$

A pair of unknown functions $\psi_K^\gamma(\tau)$ appears in (7) and is determined from the boundary conditions. We approximate the unknown functions by splines on the uniform grid

$$M_{i-1}^{\gamma,k} \frac{(t_i - t)^3}{6l} + M_i^{\gamma,k} \frac{(t - t_{i-1})^3}{6l} + \left(y_{i-1}^{\gamma,k} - M_{i-1}^{\gamma,k} \frac{l^2}{6} \right) \frac{t_i - t}{l} + \left(y_i^{\gamma,k} - M_i^{\gamma,k} \frac{l^2}{6} \right) \frac{t - t_{i-1}}{l}, \tag{9}$$

where l is the step in the variables t, $y_i^{\gamma,k}$ is the value of the function $\psi_K^\gamma(t)$ at the points $t_i = il$, and $M_i^{\gamma,k}$ are constant coefficients, related to $y_i^{\gamma,k}$ by the relations [14]

$$M_i^{\gamma,k} = -\frac{6}{l} a_{i0} y_{t,0}^{\gamma,k} - \frac{18}{l^2} \sum_{\mu=0}^{N} b_{i\mu} y_\mu^{\gamma,k} + \frac{6}{l} a_{iN} y_{t,N}^{\gamma,k}; \tag{10}$$

$b_{i\mu} = a_{i\mu} - \frac{1}{3}\delta_{i\mu}$, δ_{ii} is the Kronecker symbol, $y_{t,0}^{\gamma,k}$, $y_{t,N}^{\gamma,k}$ are the derivatives of the functions $\psi_K^\gamma(t)$ at the edges of the interval $[0, t_N]$, and a_{ij} are elements of the inverse matrix of coefficients of the system determined by a nonperiodic one-dimensional spline of type I, for whose calculation simple analytic expressions are available [14].

After substituting (9) into (7) we obtain the k-th solution

$$U_k(x, t) = S_k^1(x, t) - S_k^2(x, t) + D_k(x, t), \tag{11}$$

where the functions $S_K^\gamma(x, t)$ ($\gamma = 1, 2$) are determined for $(r-1)l < t \leqslant rl$ by the equation

$$S_k^\gamma(x, t) = \sum_{i=0}^{r} \{ M_i^{\gamma,k} l^2 K_{1,i}^{k;\gamma}(x, t) + y_i^{\gamma,k} K_{2i}^{k;\gamma}(x, t) \}. \tag{12}$$

The functions $K_{\nu,i}^{k;\gamma}(x, t)$ ($\nu = 1, 2$) depends on the interval number and on the subscript as follows: for r = 1

$$K_{1,0}^{k;\gamma}(x, t) = \frac{1}{6} \{ R_{1,1}^{\gamma,k}(x, t, t_1, t) - R_{3,1}^{\gamma,k}(x, t, t_1, t) \},$$

$$K_{1,1}^{k;\gamma}(x, t) = \frac{1}{6} \{ R_{3,1}^{\gamma,k}(x, t, t_0, t) - R_{1,1}^{\gamma,k}(x, t, t_0, t) \}, \tag{13}$$

$$K_{2,0}^{k;\gamma}(x, t) = -R_{1,1}^{\gamma,k}(x, t, t_1, t),\ K_{2,1}^{k;\gamma}(x, t) = R_{1,1}^{\gamma,k}(x, t, t_0, t), \tag{14}$$

and for $1 < r \leqslant N$

$$K_{1,0}^{k;\gamma}(x, t) = \frac{1}{6} \{ R_{1,1}^{\gamma,k}(x, t, t_1, t_1) - R_{3,1}^{\gamma,k}(x, t, t_1, t_1) \},$$

$$K_{1,i}^{k;\gamma}(x, t) = \frac{1}{6} \{ R_{3,i}^{\gamma,k}(x, t, t_{i-1}, t_i) - R_{1,i}^{\gamma,k}(x, t, t_{i-1}, t_i) + R_{1,i+1}^{\gamma,k}(x, t, t_{i+1}, t_{i+1}) - R_{3,i+1}^{\gamma,k}(x, t, t_{i+1}, t_{i+1}) \}$$

$$(i = 1, 2, \ldots, r-2), \tag{15}$$

$$K_{1,r-1}^{k;\gamma}(x, t) = \frac{1}{6} \{ R_{3,r-1}^{\gamma,k}(x, t, t_{r-2}, t_{r-1}) - R_{1,r-1}^{\gamma,k}(x, t, t_{r-2}, t_{r-1}) + R_{1,r}^{\gamma,k}(x, t, t_r, t) - R_{3,r}^{\gamma,k}(x, t, t_r, t) \},$$

$$K_{1,r}^{k;\gamma}(x, t) = \frac{1}{6} \{ R_{3,r}^{\gamma,k}(x, t, t_{r-1}, t) - R_{1,r}^{\gamma,k}(x, t, t_{r-1}, t) \},$$

$$K_{2,0}^{k;\gamma}(x, t) = -R_{1,1}^{\gamma,k}(x, t, t_1, t_1), \tag{16}$$

$$K_{2,i}^{k;\gamma}(x, t) = R_{1,i}^{\gamma,k}(x, t, t_{i-1}, t_i) - R_{1,i+1}^{\gamma,k}(x, t, t_{i+1}, t_{i+1}),$$

$$K_{2,r-1}^{k;\gamma}(x, t) = R_{1,r-1}^{\gamma,k}(x, t, t_{r-2}, t_{r-1}) - R_{1,r}^{\gamma,k}(x, t, t_r, t),$$
$$K_{2,r}^{k;\gamma}(x, t) = R_{1,r}^{\gamma,k}(x, t, t_{r-1}, t). \tag{16}$$

The following integral notation was adopted in Eqs. (13)-(16):

$$R_{m,\nu}^{\gamma,k}(x, t, q, c) = \frac{a_k}{l^m} \int_{(\nu-1)l}^{c} (\tau - q)^m \frac{\partial}{\partial \xi} G_k(x, t, H_{k-2+\gamma}, \tau)\, d\tau. \tag{17}$$

We substitute (10) into (12) and express the functions $S_k^\gamma(x, t)$ in terms of the unknown constants $y_i^{\gamma,k}$, $y_{t,0}^{\gamma,k}$, $y_{t,N}^{\gamma,k}$. Following elementary transformations, we obtain the functions $S_k^\gamma(x, t)$ in the form:

$$S_k^\gamma(x, t) = y_{t,0}^{\gamma,k} P_0^{\gamma,k}(x, t) + \sum_{i=0}^{N} y_i^{\gamma,k} L_i^{\gamma,k}(x, t) + y_{t,N}^{\gamma,k} P_N^{\gamma,k}(x, t), \tag{18}$$

where

$$P_\mu^{\gamma,k}(x, t) = (-1)^{\delta_{\mu 0}} 6l \sum_{i=0}^{r} a_{i\mu} K_{1,i}^{k,\gamma}(x, t) \quad (\mu = 0, N);$$
$$L_i^{\gamma,k}(x, t) = K_{2,i}^{k;\gamma}(x, t) \sum_{q=1}^{r} \delta_{iq} - 18 \sum_{q=1}^{r} b_{qi} K_{1,q}^{k,\gamma}(x, t). \tag{19}$$

Since $y_0^{1,k} = \varphi_k(H_{k-1})$, $y_0^{2,k} = \varphi_k(H_k)$ due to the self-consistency of the initial and boundary conditions, the total number of unknowns $y_i^{\gamma,k}$, $y_{t,0}^{\gamma,k}$, $y_{t,N}^{\gamma,k}$ for nonideal contacts equals 2n(N + 2). To determine the unknowns we use the boundary conditions (2)-(4) at times $t_i = il$ (i = 1, 2, ..., N), as well as the equations

$$y_{t,\mu}^{\gamma,k} = \frac{\partial}{\partial t} S_k^1(H_{k-2+\gamma}, t_\mu) - \frac{\partial}{\partial t} S_k^2(H_{k-2+\gamma}, t_\mu) + \frac{\partial}{\partial t} D_k(H_{k-2+\gamma}, t_\mu) \quad (\mu = 0, N), \tag{20}$$

which are obtained by differentiating (11) with respect to the variable t at the points $t = t_0$ and $t = t_N$.

We substitute (18) into (11) and (20), and then substitute (11) in the boundary conditions (2)-(4). As a result, we obtain an algebraic system of equations for the several unknowns, which in matrix form is:

$$B_0 \varkappa_0 + C_0 \varkappa_1 = d_0,$$
$$A_k \varkappa_{k-1} + B_k \varkappa_k + C_k \varkappa_{k+1} = d_k \quad (k = 1, 2, \ldots, n-1), \tag{21}$$
$$A_n \varkappa_{n-1} + B_n \varkappa_n = d_n.$$

The first and last matrix equations were obtained from boundary conditions (2), (4), in which vector unknowns consisting of N + 2 components appear:

$$\varkappa_0 = \{y_i^{1,1};\ y_{t,0}^{1,1};\ y_{t,N}^{1,1}\},\quad \varkappa_1 = \{y_i^{2,1};\ y_{t,0}^{2,1};\ y_{t,N}^{2,1}\},$$
$$\varkappa_{n-1} = \{y_i^{1,n};\ y_{t,0}^{1,n};\ y_{t,N}^{1,n}\},\quad \varkappa_n = \{y_i^{2,n};\ y_{t,0}^{2,n};\ y_{t,N}^{2,n}\}. \tag{22}$$

The elements of the matrices B_0, C_0, A_n, and B_n, being of size (N + 2) X (N + 2), are calculated by the equations:

$$b_{ij}^0 = \lambda_1 \frac{\partial}{\partial x} L_j^{1,1}(0, t_i) - \alpha_1(t_i)\delta_{ij},\quad b_{i\nu}^0 = \lambda_1 \frac{\partial}{\partial x} P_{N\delta_{\nu N+2}}^{1,1}(0, t_i),$$
$$b_{\mu j}^0 = \frac{\partial}{\partial t} L_j^{1,1}(0, t_N \delta_{\mu N+2}),\quad b_{\mu\nu}^0 = \frac{\partial}{\partial t} P_{N\delta_{\nu N+2}}^{1,1}(0, t_N \delta_{\mu N+2}) - \delta_{\mu\nu},$$
$$c_{ij}^0 = -\lambda_1 \frac{\partial}{\partial x} L_j^{2,1}(0, t_i),\quad c_{i\nu}^0 = -\lambda_1 \frac{\partial}{\partial x} P_{N\delta_{\nu N+2}}^{2,1}(0, t_i),$$
$$c_{\mu j}^0 = -\frac{\partial}{\partial t} L_j^{2,1}(0, t_N \delta_{\mu N+2}),\quad c_{\mu\nu}^0 = -\frac{\partial}{\partial t} P_{N\delta_{\nu N+2}}^{2,1}(0, t_N \delta_{\mu N+2}), \tag{23}$$

$$a^n_{ij} = -\lambda_n \frac{\partial}{\partial x} L^{1,n}(H_n,\ t_i),\quad a^n_{i\nu} = -\lambda_n \frac{\partial}{\partial x} P^{1,n}_{N\delta_{\nu N+2}}(H_n,\ t_i),$$

$$a^n_{\mu j} = -\frac{\partial}{\partial t} L^{1,n}_j(H_n,\ t_N\delta_{\mu N+2}),\quad a^n_{\mu\nu} = -\frac{\partial}{\partial t} P^{1,n}_{N\delta_{\nu N+2}}(H_n,\ t_N\delta_{\mu N+2}),$$

$$b^n_{ij} = \lambda_n \frac{\partial}{\partial x} L^{2,n}_j(H_n,\ t_i) - \alpha_2(t_i)\delta_{ij},\quad b^n_{i\nu} = \lambda_n \frac{\partial}{\partial x} P^{2,n}_{N\delta_{\nu N+2}}(H_n,\ t_i),$$

$$b^n_{\mu j} = \frac{\partial}{\partial t} L^{2,n}_j(H_n,\ t_N\delta_{\mu N+2}),\quad b^n_{\mu\nu} = \frac{\partial}{\partial t} P^{2,n}_{N\delta_{\nu N+2}}(H_n,\ t_N\delta_{\mu N+2}) + \delta_{\mu\nu}.$$

The free term vectors consist of the following elements:

$$d^0_i = \lambda_1\left\{\varphi_1(H_1)\frac{\partial}{\partial x}L^{2,1}_0(0,\ t_i) - \varphi_1(0)\frac{\partial}{\partial x}L^{1,1}_0(0,\ t_i) - \frac{\partial}{\partial x}D_1(0,\ t_i)\right\} - f_1(t_i),$$

$$d^0_\mu = \varphi_1(H_1)\frac{\partial}{\partial t}L^{2,1}_0(0,\ t_N\delta_{\mu N+2}) - \varphi_1(0)\frac{\partial}{\partial t}L^{1,1}_0(0,\ t_N\delta_{\mu N+2}) - \frac{\partial}{\partial t}D_1(0,\ t_N\delta_{\mu N+2}),$$

$$d^n_i = \lambda_n\left\{\varphi_n(H_{n-1})\frac{\partial}{\partial x}L^{1,n}_0(H_n,\ t_i) - \varphi_n(H_n)\frac{\partial}{\partial x}L^{2,n}_0(H_n,\ t_i) + \frac{\partial}{\partial x}D_n(H_n,\ t_i)\right\} - f_2(t_i), \tag{24}$$

$$d^n_\mu = \frac{\partial}{\partial t}D_n(H_n,\ t_N\delta_{\mu N+2}) + \varphi_n(H_{n-1})\frac{\partial}{\partial t}L^{1,n}_0(H_n,\ t_N\delta_{\mu N+2}) - \varphi_n(H_n)\frac{\partial}{\partial t}L^{2,n}_0(H_n,\ t_N\delta_{\mu N+2}).$$

The unknown vectors appearing in the matrix equations, obtained on the k-th inner boundary (k = 1, 2, ..., n − 1), contain the following elements:

$$\varkappa_{k-1} = \{y^{1,k}_i;\ y^{1,k}_{t,0};\ y^{1,k}_{t,N}\},\quad \varkappa_{k-1} = \{y^{2,k+1}_i;\ y^{2,k+1}_{t,0};\ y^{2,k+1}_{t,N}\},$$
$$\varkappa_k = \{y^{2,k}_i;\ y^{2,k}_{t,0};\ y^{2,k}_{t,N};\ y^{1,k+1}_i;\ y^{1,k+1}_{t,0};\ y^{1,k+1}_{t,N}\}. \tag{25}$$

The caged matrices A_k, B_k, and C_k have the structure:

$$A_k = \begin{bmatrix} A^1_k \\ A^2_k \end{bmatrix},\quad B_k = \begin{bmatrix} B^1_k, & B^1_{k+1} \\ B^2_k, & B^2_{k+1} \end{bmatrix},\quad C_k = \begin{bmatrix} C^1_k \\ C^2_k \end{bmatrix}, \tag{26}$$

where the elements of the inner matrices, having size (N+ 2) X (N + 2), are calculated by the equations:

$$a^{1,k}_{ij} = -\lambda_k\frac{\partial}{\partial x}L^{1,k}_j(H_k,\ t_i),\quad a^{1,k}_{i\nu} = -\lambda_k\frac{\partial}{\partial x}P^{1,k}_{N\delta_{\nu N+2}}(H_k,\ t_i),$$

$$a^{1,k}_{\mu j} = -\frac{\partial}{\partial t}L^{1,k}_j(H_k,\ t_N\delta_{\mu N+2}),\quad a^{1,k}_{\mu\nu} = -\frac{\partial}{\partial t}P^{1,k}_{N\delta_{\nu N+2}}(H_k,\ t_N\delta_{\mu N+2}),$$

$$a^{2,k}_{ij} = R_k a^{1,k}_{ij},\quad a^{2,k}_{\mu j} = 0\quad (j = 1,\ 2,\ \ldots,\ N+2), \tag{27}$$

$$b^{1,k}_{ij} = \lambda_k\frac{\partial}{\partial x}L^{2,k}_j(H_k,\ t_i),\quad b^{1,k}_{i\nu} = \lambda_k\frac{\partial}{\partial x}P^{2,k}_{N\delta_{\nu N+2}}(H_k,\ t_i),$$

$$b^{1,k}_{\mu j} = \frac{\partial}{\partial t}L^{2,k}_j(H_k,\ t_N\delta_{\mu N+2}),\quad b^{1,k}_{\mu\nu} = \frac{\partial}{\partial t}P^{2,k}_{N\delta_{\nu N+2}}(H_k,\ t_N\delta_{\mu N+2}) + \delta_{\mu\nu},$$

$$b^{2,k}_{ij} = R_k b^{1,k}_{ij} + \delta_{ij},\quad b^{2,k}_{i\nu} = R_k b^{1,k}_{i\nu},\quad b^{2,k}_{\mu j} = 0\quad (j = 1,\ 2,\ \ldots,\ N+2),$$

$$b^{1,k+1}_{ij} = \lambda_{k+1}\frac{\partial}{\partial x}L^{1,k+1}_j(H_k,\ t_i),\quad b^{1,k+1}_{i\nu} = \lambda_{k+1}\frac{\partial}{\partial x}P^{1,k+1}_{N\delta_{\nu N+2}}(H_k,\ t_i),\quad b^{1,k+1}_{\mu j} = 0,$$

$$b^{2,k+1}_{ij} = -\delta_{ij},\quad b^{2,k+1}_{i\nu} = 0,$$

$$b^{2,k+1}_{\mu j} = \frac{\partial}{\partial t}L^{1,k+1}_j(H_k,\ t_N\delta_{\mu N+2}),\quad b^{2,k+1}_{\mu\nu} = \frac{\partial}{\partial t}P^{1,k+1}_{N\delta_{\nu N+2}}(H_k,\ t_N\delta_{\mu N+2}) - \delta_{\mu\nu},$$

$$c_{ij}^{1,k} = -\lambda_{k+1}\frac{\partial}{\partial x}L_j^{2,k+1}(H_k, t_i),\ c_{i\nu}^{1,k} = -\lambda_{k+1}\frac{\partial}{\partial x}P_{N\delta_{\nu N+2}}^{2,k+1}(H_k, t_i),$$

$$c_{\mu j}^{1,k} = 0,\quad c_{ij}^{2,k} = 0 \quad (j = 1, 2, \ldots, N+2),$$

$$c_{\mu j}^{2,k} = -\frac{\partial}{\partial t}L_j^{2,k+1}(H_k, t_N\delta_{\mu N+2}),\ c_{\mu\nu}^{2,k} = -\frac{\partial}{\partial t}P_{N\delta_{\nu N+2}}^{2,k+1}(H_k, t_N\delta_{\mu N+2}).$$

The elements of the vector $d_k = [d_k^1, d_k^2]$ are calculated by the equations:

$$d_i^{1,k} = \lambda_k\left\{\varphi_k(H_{k-1})\frac{\partial}{\partial x}L_0^{1,k}(H_k, t_i) - \varphi_k(H_k)\frac{\partial}{\partial x}L_0^{2,k}(H_k, t_i) + \right.$$

$$\left. + \frac{\partial}{\partial x}D_k(H_k, t_i)\right\} - \lambda_{k+1}\left\{\varphi_{k+1}(H_k)\frac{\partial}{\partial x}L_0^{1,k+1}(H_k, t_i) - \varphi_{k+1}(H_{k+1})\frac{\partial}{\partial x}L_0^{2,k+1}(H_k, t_i) + \frac{\partial}{\partial x}D_{k+1}(H_k, t_i)\right\},$$

$$d_\mu^{1,k} = \frac{\partial}{\partial t}D_k(H_k, t_N\delta_{\mu N+2}) + \varphi_k(H_{k-1})\frac{\partial}{\partial t}L_0^{1,k}(H_k, t_N\delta_{\mu N+2}) - \varphi_k(H_k)\frac{\partial}{\partial t}L_0^{2,k}(H_k, t_N\delta_{\mu N+2}),$$

$$d_i^{2,k} = R_k\lambda_k\left\{\varphi_k(H_{k-1})\frac{\partial}{\partial x}L_0^{1,k}(H_k, t_i) - \varphi_k(H_k)\frac{\partial}{\partial x}L_0^{2,k}(H_k, t_i) + \frac{\partial}{\partial x}D_k(H_k, t_i)\right\}, \tag{28}$$

$$d_\mu^{2,k} = \varphi_{k+1}(H_{k+1})\frac{\partial}{\partial t}L_0^{2,k+1}(H_k, t_N\delta_{\mu N+2}) - \varphi_{k+1}(H_k)\frac{\partial}{\partial t}L_0^{1,k+1}(H_k, t_N\delta_{\mu N+2}) - \frac{\partial}{\partial t}D_{k+1}(H_k, t_N\delta_{\mu N+2}).$$

The subscripts i, j in Eqs. (23)-(28) acquire the values from 1 to N (except for especially marked cases), and the subscripts μ, ν vary from N + 1 to N + 2.

If at the k-th boundary $R_k = 0$ (ideal contact), it follows from (3) that

$$\psi_k^2(t) = \psi_{k+1}^1(t). \tag{29}$$

Hence the number of unknowns in the k-th matrix equation (21) is decreased by N + 2, i.e., $y_i^{2,k} = y_i^{1,k+1} = y_i^k$; $y_{t,q}^{2,k} = y_{t,q}^{1,k+1} = y_{t,q}^k$ $(q = 0, N)$, and the vector $\varkappa_k$ is for ideal contact

$$\varkappa_k = \{y_i^k;\ y_{t,0}^k;\ y_{t,N}^k\}. \tag{30}$$

The elements of the square matrices A_k, B_k, and C_k and of the vector d_k are calculated for ideal contact by the equations:

$$a_{ij}^k = a_{ij}^{1,k},\ b_{ij}^k = b_{ij}^{1,k} + b_{ij}^{1,k+1},\ c_{ij}^k = c_{ij}^{1,k},\ d_i^k = d_i^{1,k}\ \begin{pmatrix} i=1, 2, \ldots, N+2 \\ j=1, 2, \ldots, N+2 \end{pmatrix}.$$

If boundary conditions of the first kind are given on the outer boundaries of the multilayer body

$$U_1|_{x=0} = f_1(t);\ U_n|_{x=H_n} = f_2(t), \tag{31}$$

it then follows from (6) that

$$\psi_1^1(t) = f_1(t),\ \psi_n^2(t) = f_2(t). \tag{32}$$

In this case the first and last matrix equations in (21) are absent, the matrices A_1 and C_{n-1} are vanishing matrices, and in the elements of the free term vectors d_1 and d_{n-1} one must insert in the corresponding places the unknown functions $S^1(H_1, t)$ and $S_n^2(H_{n-1}, t)$ at the required moments of time.

Thus, the original boundary-value problem (1)-(5) was reduced to a system of algebraic equations (21), whose structure remains invariant both for arbitrary boundary conditions at the outer boundaries and for various amounts of contact on the boundaries of the compound layers. Only the number of matrix equations of system (21) changes as a function of the shape of boundary conditions, and the nature of contact at the k-th boundary changes only the dimensionality of matrices of the k-th equation.

We turn to the solution of the algebraic system (21), whose coefficient matrix has a caged tridiagonal structure, which makes it possible to suggest a quite effective algorithm for its solution.

We calculate the auxiliary matrices

$$P_k = A_k Q_{k-1} + B_k,\ Q_{-1} = 0\ (k = 0,\ 1,\ \ldots,\ n),$$
$$Q_k = -P_k^{-1} C_k \quad (k = 0,\ 1,\ \ldots,\ n-1), \tag{33}$$
$$U_k = P_k^{-1}(d_k - A_k U_{k-1}),\ U_{-1} = 0\ (k = 0,\ 1,\ \ldots,\ n).$$

Successively eliminating the vectors $\varkappa_0, \ldots, \varkappa_{n-1}$ from the first, ..., (n + 1)-th equations, we obtain an equivalent matrix system

$$\varkappa_k = Q_k \varkappa_{k+1} + U_k\ (k = 1,\ 2,\ \ldots,\ n-1),\ \varkappa_n = U_n, \tag{34}$$

from which the unknowns are easily determined by recurrence. Equations (34) are proved by mathematical induction.

If boundary conditions of the first kind are given at x = 0, the subscript k acquires values starting from 1, and the matrices Q_0 and U_0 are vanishing. For a given boundary condition of the first kind at the lower boundary the subscript k varies till n − 1.

Theorems were proved in [14], verifying that a cubic spline and its derivative converge uniformly to a continuous function and its derivative when the norm of the grid tends to zero. Similar theorems were established for two-dimensional functions when approximated by twofold cubic splines.

Starting from this, we approximate the initial distribution $\varphi_k(x)$ by a one-dimensional spline. On the interval $[(j-1)h_k^*, jh_k^*]$ the initial distribution is then

$$M_{j-1}^{\varphi_k} \frac{(x_j - x)^3}{6h_k^*} + M_j^{\varphi_k} \frac{(x - x_{j-1})^3}{6h_k^*} + \left(\varphi_{j-1}^k - M_{j-1}^{\varphi_k} \frac{(h_k^*)^2}{6}\right)\frac{x_j - x}{h_k^*} + \left(\varphi_j^k - M_j^{\varphi_k} \frac{(h_k^*)^2}{6}\right)\frac{x - x_{j-1}}{h_k^*}, \tag{35}$$

where φ_j^k are values of the functions $\varphi_k(x)$ at the points $x_j = jh_k^*$; $h_k^* = h_k/M_k$; M_k is the partition number of the region $[H_{k-1}, H_k]$, and $M_j^{\varphi k}$ are known constants.

We substitute (35) into (8) and transform. As a result we obtain

$$D_k(x,\ t) = \sum_{j=0}^{M_k} \{M_j^{\varphi_k} (h_k^*)^2 N_{1,j}^k (x,\ t) + \varphi_j^k N_{2,j}^k (x,\ t)\}, \tag{36}$$

where

$$N_{1,0}^k (x,\ t) = \frac{1}{6}\{Q_1^{k,1}(x,\ t,\ x_1) - Q_1^{k,3}(x,\ t,\ x_1)\};$$
$$N_{1,j}^k (x,\ t) = \frac{1}{6}\{Q_j^{k,3}(x,\ t,\ x_{j-1}) - Q_j^{k,1}(x,\ t,\ x_{j-1}) + Q_{j+1}^{k,1}(x,\ t,\ x_{j+1}) - Q_j^{k,3}(x,\ t,\ x_{j+1})\} \quad (j = 1,\ 2,\ \ldots,\ M_k - 1);$$
$$N_{1,M_k}^k (x,\ t) = \frac{1}{6}\{Q_{M_k}^{k,3}(x,\ t,\ x_{M_k - 1}) - Q_{M_k}^{k,1}(x,\ t,\ x_{M_k - 1})\}; \tag{37}$$
$$N_{2,0}^k (x,\ t) = -Q_1^{k,1}(x,\ t,\ x_1);$$
$$N_{2,j}^k (x,\ t) = Q_j^{k,1}(x,\ t,\ x_{j-1}) - Q_{j+1}^{k,1}(x,\ t,\ x_{j+1}) \quad (j = 1,\ 2,\ \ldots,\ M_k - 1);$$
$$N_{2,M_k}^k (x,\ t) = Q_{M_k}^{k,1}(x,\ t,\ x_{M_k - 1}).$$

The following integral notation was adopted in Eqs. (37):

$$Q_j^{k,m}(x,\ t,\ q) = \frac{1}{(h_k^*)^m} \int_{(j-1)h_k^* + H_{k-1}}^{jh_k^* + H_{k-1}} (\xi - q)^m G_k(x,\ t,\ \xi,\ 0)\, d\xi. \tag{38}$$

If a heat source $F_k(x, t)$ is found in the k-th layer, the function $D_k(x, t)$ is supplemented by the term

$$\int_0^t \int_{H_{k-1}}^{H_k} F_k(\xi,\ \tau)\, G_k(x,\ t,\ \xi,\ \tau)\, d\xi d\tau,$$

whose calculation is conveniently carried out by approximating the function $F_k(x, t)$ by twofold cubic splines. In this case the dominent behavior of the solution remains unchanged, while the elements of the free term vector in the k-th layer have an additional term.

Consider the method of calculating the temperature field in composite bodies on the basis of the equations obtained. Let it be required to find the solution of the boundary-value problem (1)-(5) on the time interval [0, t*]. We partition the given interval into equal segments $\nu t_N (\nu = 1, 2, \ldots)$, and then assign a uniform grid $t_0 < t_1 \ldots < t_N$ with step l on each ν-th segment. The step l is chosen from the accuracy condition of the approximation of the assigned functions $f_1(t)$ and $f_2(t)$, and the number N is selected taking into account the memory of a given computer. By the equations derived we find the elements of the matrices A_k, B_k, C_k. To calculate the auxiliary matrices P_k and Q_k (33) it is necessary to find n inverse matrices of size (N + 2) × (N + 2) in the case of ideal contact, or 2(N + 2) × 2(N + 2) in case of nonideal contact. Since for boundary conditions of the first and second kinds, as well as for boundary conditions of the third kind in the case of periodic functions $\alpha_1(t)$ and $\alpha_2(t)$ on each ν-th interval the matrices A_k, B_k, C_k remain invariant (23), the procedure of finding the inverse matrices is accomplished only once. If $\alpha_2(t)$ is a nonperiodic function, at each ν-th segment one must recalculate only the matrix P_n. The algorithm of finding the unknown vectors $\varkappa_k$ at the ν-th segment reduces to a simple calculation of the matrices U_k (33) and a recurrent calculation of $\varkappa_k$ by Eqs. (34). The values of the unknown functions $U_k(x, t)$ are found from Eqs. (11), (18), and (8). Since the functions $L_i^{\gamma,k}(x, t)$ and $P_\mu^{\gamma,k}(x, t)$ are independent of $y_i^{\gamma,k}$, $y_{t,q}^{\gamma,k}$, their calculation is also performed once during the whole calculation. After finding the points $U_k(x_j, t_N)$ at the ν-th segment the coefficients $M_i^{\varphi}k$ are calculated for the initial distribution at the $(\nu + 1)$-th step, and the whole procedure is repeated.

Since the calculation of the functions and finding the inverse matrices during the calculation occurs only once, the algorithm suggested is quite effective for calculating the temperature field in composite bodies with large t* values.

The method suggested has been used so far to solve linear problems. In a following publication we intend to show how the method can be extended to a class of nonlinear problems.

NOTATION

$U_k \equiv U_k(x, t)$, temperature of the k-th layer; h_k, width of the k-th layer; $H_k = \sum_{j=1}^{k} h_j$; $F_k(x, t)$, a prescribed source function; λ_k, a_k, thermal conductivity and thermal diffusivity of the k-th layer; R_k, thermal surface resistance coefficient; $\alpha_1(t)$, $\alpha_2(t)$, heat-transfer coefficients at the external boundaries of a composite body; $\varphi_k(x)$, initial temperature distribution of the k-th layer; and $f_1(t)$, $f_2(t)$, prescribed time functions.

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